MSUCP-12

MICHIGAN STATE UNIVERSITY

CYCLOTRON PROJECT*

Computation of Electric Field and Potential of an Idealized Dee Geometry

J. W. Beal

October 1961

Department of Physics

East Lansing, Michigan

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Introduction:

This paper presents in summary formulas for the computation of electric fields and potentials of an idealized cyclotron dee geometry. A method is presented for calculation of the median plane fields and potentials. This median plane method was utilized in a computer program, and results are presented for various dee geometry arrangements.

I. General Results:

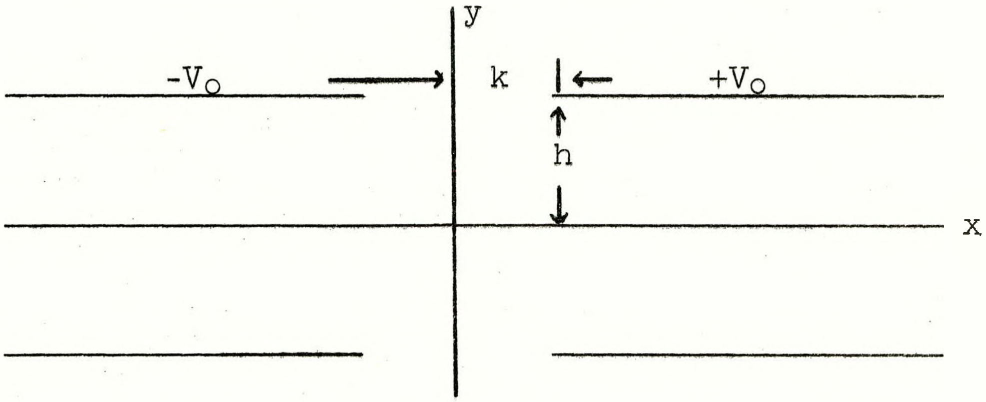

Fig. 1 Cyclotron Dee Geometry

The analytic solutions for E and V were determined for the idealized cyclotron dee geometry as shown in Fig. 1. Following the method outlined in the article "Electric Fields Within Cyclotron Dees"[1] with annotations and corrections,[2] the Schwarz - Christoffel transformation was used to map the dee geometry to one for which the electric field and potential was known. The complex equation $x + iy = f(U + iV)$ was obtained.

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1) R.L. Murray and L.T. Ratner, Journal of Applied Physics, 24, 67, (1953)

2) R.L. Murray, private communication

$$x+iy = \frac{2h}{\pi}\left[\log(\tanh s + i\,\mathrm{sech}\, s) - i\frac{1}{A}\sinh s - \frac{i\pi}{2}\right] \qquad 1)$$

where, $s = u+iv$ $\qquad u = \frac{\pi U}{2V_o}$; $\qquad v = \frac{\pi V}{2V_o}$

and $\frac{\pi}{2}\frac{k}{h} = \cosh^{-1}\left(\frac{1}{a}\right) + \frac{(1-a^2)^{1/2}}{a^2}$; $\qquad A = \frac{a^2}{1-a^2}$

The electric fields of interest are;

$$E_x = \frac{\partial V}{\partial x} ; \qquad E_y = \frac{\partial V}{\partial y}$$

Following Murray and Ratner letting $Y = \frac{\pi y}{2h}$; $X = \frac{\pi x}{2h}$;

$$E_x = \frac{V_o}{h}\left[\frac{X_v}{(X_v^2+X_u^2)}\right] \qquad 2)$$

$$E_y = \frac{V_o}{h}\left[\frac{X_u}{(X_v^2+X_u^2)}\right]$$

The derivatives X_v and X_u are determined:

$$X_v = \cosh X_1 \cos Y_1 \left(1 + \frac{1}{AF^2}\right) \qquad 3)$$

$$X_u = -\sinh X_1 \sin Y_1 \left(1 - \frac{1}{AF^2}\right)$$

$$F = [\cosh^2 X_1 - \sin^2 Y_1]^{1/2}$$

where X_1 and Y_1 are determined by iteration described in the following section.

The potential may be determined by:

$$v = \cos^{-1}\left[\frac{\cos Y_1}{F}\right] \tag{4}$$

II. Computation Methods:

In the formulas given above the electric field and potential are explicit functions of the variable X_1 and Y_1 rather than X and Y. Therefore the quantities X_1 and Y_1 must be determined for a given X and Y from the transcendental equations:

$$X = X_1 + \frac{\sinh X_1 \cosh X_1}{AF^2} \tag{5}$$

$$-Y = Y_1 + \frac{\sin Y_1 \cos Y_1}{AF^2}$$

Therefore, equations 2 and 4 coupled with equations 3 and 5 can be used to determine the electric field and potential at a point X, Y of the dee region.

III. Median Plane Studies:

For the present, it was desired to study the fields and potentials in the median plane. This specification considerably simplifies the above equations:

$$E_x = \frac{V_o}{h}\left[\frac{\operatorname{sech} X_1}{1+\frac{1}{A}\operatorname{sech}^2 X_1}\right] \qquad E_x(x) = E_x(-x) \tag{6}$$

$$E_y \equiv 0$$

$$v = \cos^{-1}[\operatorname{sech} X_1] \qquad v(x) = -v(-x) \tag{7}$$

$$X = X_1 + \alpha \tanh X_1 \tag{8}$$

$$Y \equiv 0$$

$$\alpha \equiv \frac{1}{A}$$

These above equations are well behaved and simpler to work with than the general equations.

The iteration process suggested by Murray and Ratner for calculation of X_1 converges slowly for small values of α, and does not converge at all for large α. An iteration process, described below, suggested by M.M. Gordon was used and was found to work well for all cases.

Assume that X_1^* is an approximate solution to equation 8 such that $(X_1 - X_1^*)$ is small. Expanding tanh X_1 in equation 8 to first-order in $(X_1 - X_1^*)$, and solving for X_1, the following result is obtained:

$$X_1 = \frac{X - \alpha \tanh X_1^* + \alpha X_1^* \operatorname{sech}^2 X_1^*}{1 + \alpha \operatorname{sech}^2 X_1^*} \tag{9}$$

The iteration process then consists of taking an initial guess for X_1 setting this equal to X_1^* on the right and solving X_1; then taking this new X_1 and setting it equal to X_1^* on the right, again solve for X_1; repeating this process until the difference in successive values of X_1 is sufficiently small. This iteration is Newtonian in character such that if a given X_1^* has an error of order ϵ, then X_1 given by equation 9 will have an error at order ϵ^2.

In order for the iteration process to work properly, it is necessary to determine a reasonable guess for the initial value of X_1. To obtain such a guess it need only be noted since $\tanh X_1 \lesssim X_1$ for small X_1 then

$$X_1 \gtrsim X/(1+\alpha)$$

for X_1 small, and since $\tanh X_1 \lesssim 1$ for large X_1, then

$$X_1 \gtrsim X-\alpha$$

for X_1 large. A simple choice for the initial value of X_1 is then

$$X_1 = X/(1+\alpha) \quad \text{for } (1+\alpha) > X$$

$$X_1 = X-\alpha \quad \text{for } (1+\alpha) \leq X$$

With this choice, the iteration procedure given above works well for all values of α.

IV. Results -

A fixed point computer program written by D.A. Johnson for use on MISTIC was used to calculate the above equations for E_x and V_x at the point $(X,0)$ with $\alpha<4$. Figure 2 shows the electric field for several gap parameters, and figure 3 shows the potential for these same dee geometrics. Table 1 gives values of electric field and potential for a wide range of dee gap arrangements. Interested persons desiring values of E_x and V_x for gap arrangements not listed in the table may gain such information by writing to this laboratory.

Appreciation is extended to M.M. Gordon of this department for his helpful discussions and suggestions, and to D.A. Johnson for his work on computer studies.

Fig. 2

$\frac{E_x}{V_o/h}$ v.s. $\frac{x}{h}$

symbol ▲ represent: $x = k$

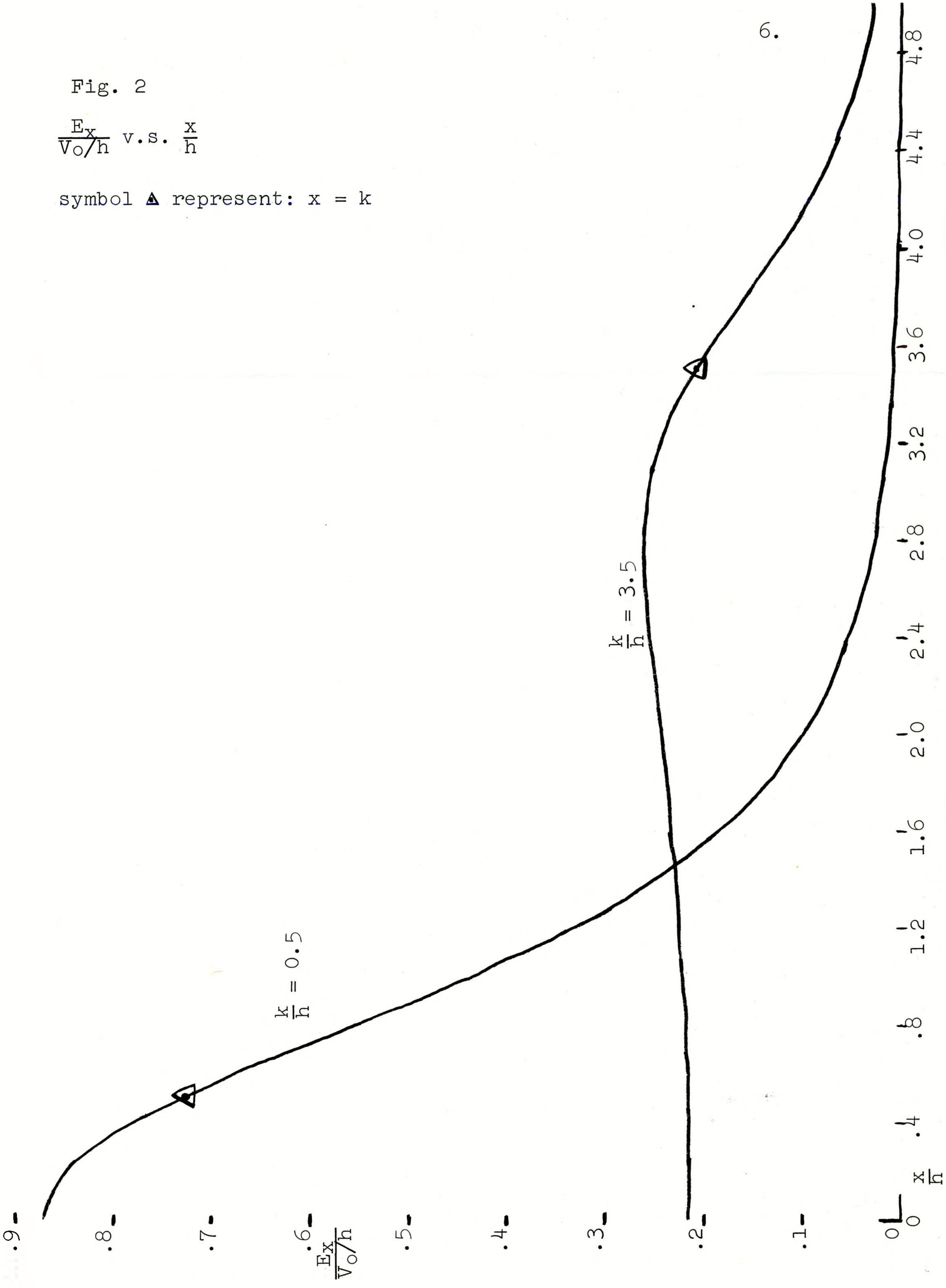

Fig. 3

$\frac{V_x}{V_o}$ vs. $\frac{x}{h}$

symbol ▲ represents x = k

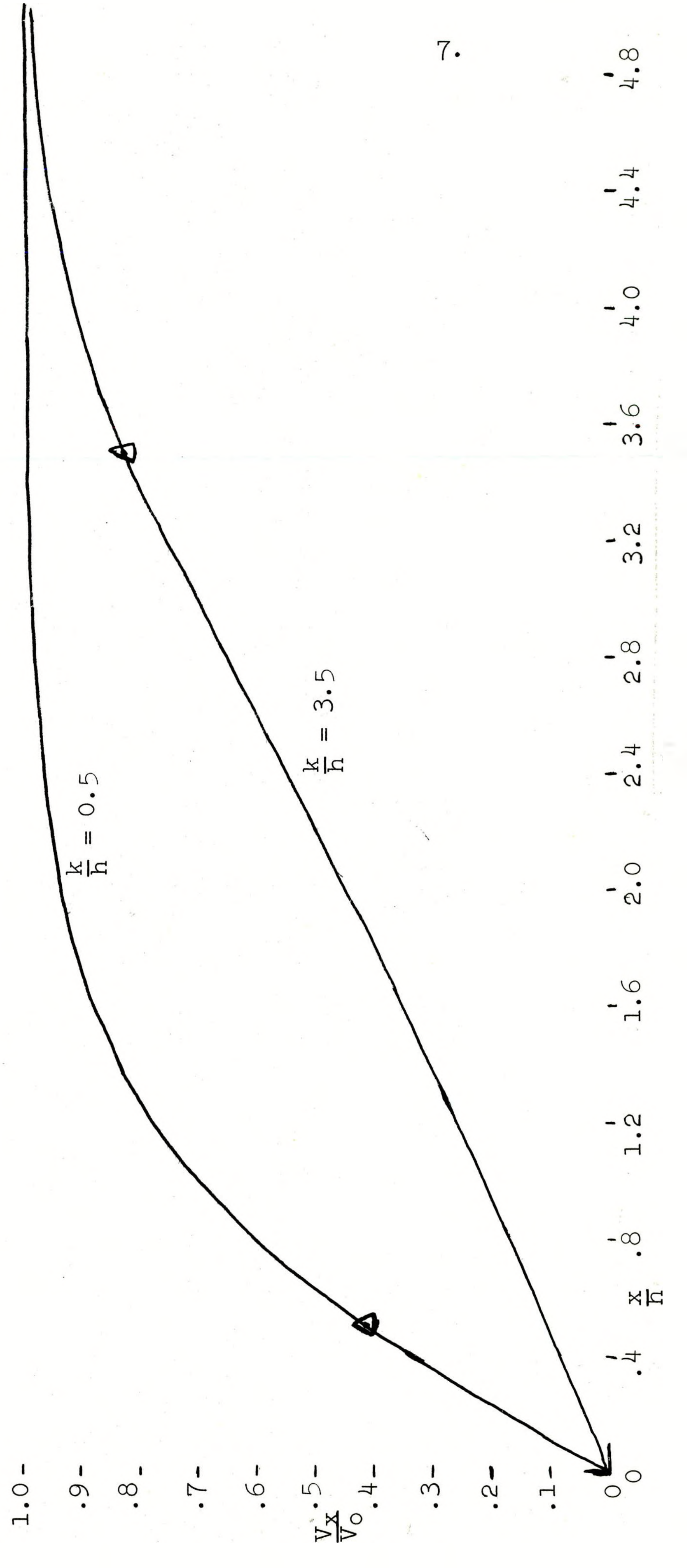

Table 1.

$\frac{k}{h} = 0.1$

$\frac{x}{h}$	$\frac{E}{V_o/h}$	$\frac{V}{V_o}$
0.00	0.99388	0.00000
0.20	0.94784	0.19566
0.40	0.82863	0.37420
0.60	0.67692	0.52496
0.80	0.52800	0.64520
1.00	0.40021	0.73760
1.20	0.29834	0.80702
1.40	0.22033	0.85853
1.60	0.16189	0.89647
1.80	0.11862	0.92430
2.00	0.08679	0.94468
2.20	0.06345	0.95958
2.40	0.04637	0.97047
2.60	0.03388	0.97843
2.80	0.02475	0.98424
3.00	0.01808	0.98849
3.20	0.01320	0.99159
3.40	0.00964	0.99386
3.60	0.00704	0.99551
3.80	0.00515	0.99672
4.00	0.00376	0.99760
4.20	0.00274	0.99825
4.40	0.00200	0.99872
4.60	0.00146	0.99906
4.80	0.00107	0.99931
5.00	0.00078	0.99950

$\frac{k}{h} = 0.3$

$\frac{x}{h}$	$\frac{E}{V_o/h}$	$\frac{V}{V_o}$
0.00	0.94834	0.00000
0.20	0.91160	0.18719
0.40	0.81244	0.36044
0.60	0.67821	0.50983
0.80	0.53843	0.63138
1.00	0.41305	0.72618
1.20	0.31019	0.79811
1.40	0.23005	0.85178
1.60	0.16943	0.89144
1.80	0.12431	0.92059
2.00	0.09101	0.94196
2.20	0.06656	0.95759
2.40	0.04865	0.96902
2.60	0.03555	0.97737
2.80	0.02597	0.98347
3.00	0.01897	0.98792
3.20	0.01386	0.99118
3.40	0.01012	0.99355
3.60	0.00739	0.99529
3.80	0.00540	0.99656
4.00	0.00394	0.99748
4.20	0.00288	0.99816
4.40	0.00210	0.99866
4.60	0.00154	0.99902
4.80	0.00112	0.99928
5.00	0.00082	0.99947

$\frac{k}{h} = 0.5$

$\frac{x}{h}$	$\frac{E}{V_o/h}$	$\frac{V}{V_o}$
0.00	0.87049	0.00000
0.20	0.84658	0.17250
0.40	0.77779	0.33562
0.60	0.67446	0.48129
0.80	0.55436	0.60429
1.00	0.43639	0.70319
1.20	0.33326	0.77985
1.40	0.24966	0.83781
1.60	0.18492	0.88098
1.80	0.13610	0.91285
2.00	0.09982	0.93626
2.20	0.07307	0.95341
2.40	0.05343	0.96596
2.60	0.03905	0.97513
2.80	0.02853	0.98183
3.00	0.02084	0.98673
3.20	0.01523	0.99030
3.40	0.01112	0.99292
3.60	0.00812	0.99483
3.80	0.00593	0.99622
4.00	0.00433	0.99724
4.20	0.00317	0.99798
4.40	0.00231	0.99852
4.60	0.00169	0.99892
4.80	0.00123	0.99921
5.00	0.00090	0.99942

$\frac{k}{h} = 1.0$

$\frac{x}{h}$	$\frac{E}{V_o/h}$	$\frac{V}{V_o}$
0.00	0.65448	0.00000
0.20	0.65008	0.13061
0.40	0.63538	0.25935
0.60	0.60647	0.38381
0.80	0.55908	0.50070
1.00	0.49250	0.60615
1.20	0.41267	0.69680
1.40	0.33033	0.77105
1.60	0.25529	0.82943
1.80	0.19269	0.87401
2.00	0.14336	0.90740
2.20	0.10577	0.93214
2.40	0.07768	0.95034
2.60	0.05690	0.96370
2.80	0.04163	0.97347
3.00	0.03043	0.98062
3.20	0.02224	0.98584
3.40	0.01625	0.98965
3.60	0.01187	0.99244
3.80	0.00867	0.99448
4.00	0.00633	0.99597
4.20	0.00462	0.99705
4.40	0.00338	0.99785
4.60	0.00247	0.99842
4.80	0.00180	0.99885
5.00	0.00132	0.99916

$\frac{k}{h} = 1.5$

$\frac{x}{h}$	$\frac{E}{V_o/h}$	$\frac{V}{V_o}$
0.00	0.48916	0.00000
0.20	0.48925	0.09784
0.40	0.48911	0.19569
0.60	0.48737	0.29338
0.80	0.48150	0.39036
1.00	0.46770	0.48545
1.20	0.44151	0.57662
1.40	0.40000	0.66103
1.60	0.34478	0.73568
1.80	0.28269	0.79847
2.00	0.22231	0.84888
2.20	0.16970	0.88793
2.40	0.12711	0.91744
2.60	0.09414	0.93942
2.80	0.06928	0.95564
3.00	0.05081	0.96755
3.20	0.03719	0.97628
3.40	0.02720	0.98267
3.60	0.01988	0.98734
3.80	0.01452	0.99075
4.00	0.01061	0.99324
4.20	0.00775	0.99506
4.40	0.00566	0.99639
4.60	0.00414	0.99736
4.80	0.00302	0.99807
5.00	0.00221	0.99859

$\frac{k}{h} = 2.0$

$\frac{x}{h}$	$\frac{E}{V_o/h}$	$\frac{V}{V_o}$
0.00	0.37823	0.00000
0.20	0.37887	0.07569
0.40	0.38072	0.15163
0.60	0.38353	0.22804
0.80	0.38674	0.30507
1.00	0.38936	0.38270
1.20	0.38966	0.46066
1.40	0.38492	0.53823
1.60	0.37151	0.61405
1.80	0.34596	0.68602
2.00	0.30731	0.75155
2.20	0.25912	0.80830
2.40	0.20831	0.85502
2.60	0.16142	0.89189
2.80	0.12203	0.92010
3.00	0.09087	0.94126
3.20	0.06708	0.95695
3.40	0.04928	0.96849
3.60	0.03610	0.97696
3.80	0.02641	0.98316
4.00	0.01931	0.98770
4.20	0.01411	0.99101
4.40	0.01031	0.99343
4.60	0.00753	0.99520
4.80	0.00550	0.99649
5.00	0.00402	0.99744

$\frac{k}{h} = 2.5$

$\frac{x}{h}$	$\frac{E}{V_o/h}$	$\frac{V}{V_o}$
0.00	0.30581	0.00000
0.20	0.30636	0.06120
0.40	0.30799	0.12262
0.60	0.31067	0.18446
0.80	0.31431	0.24694
1.00	0.31872	0.31024
1.20	0.32357	0.37446
1.40	0.32818	0.43965
1.60	0.33137	0.50565
1.80	0.33119	0.57198
2.00	0.32478	0.63771
2.20	0.30885	0.70125
2.40	0.28125	0.76046
2.60	0.24321	0.81304
2.80	0.19978	0.85738
3.00	0.15726	0.89302
3.20	0.12011	0.92065
3.40	0.08998	0.94154
3.60	0.06665	0.95710
3.80	0.04906	0.96859
4.00	0.03598	0.97702
4.20	0.02634	0.98320
4.40	0.01926	0.98773
4.60	0.01408	0.99103
4.80	0.01029	0.99345
5.00	0.00751	0.99521

$\frac{k}{h} = 3.0$

$\frac{x}{h}$	$\frac{E}{V_o/h}$	$\frac{V}{V_o}$
0.00	0.25301	0.00000
0.20	0.25341	0.05063
0.40	0.25459	0.10142
0.60	0.25658	0.15252
0.80	0.25936	0.20410
1.00	0.26293	0.25631
1.20	0.26727	0.30932
1.40	0.27226	0.36326
1.60	0.27772	0.41826
1.80	0.28324	0.47436
2.00	0.28805	0.53151
2.20	0.29084	0.58945
2.40	0.28950	0.64758
2.60	0.28119	0.70478
2.80	0.26320	0.75940
3.00	0.23470	0.80935
3.20	0.19844	0.85275
3.40	0.15983	0.88858
3.60	0.12399	0.91688
3.80	0.09380	0.93856
4.00	0.06987	0.95482
4.20	0.05159	0.96688
4.40	0.03790	0.97576
4.60	0.02777	0.98228
4.80	0.02032	0.98705
5.00	0.01485	0.99054

$\frac{k}{h} = 3.5$

$\frac{x}{h}$	$\frac{E}{V_o/h}$	$\frac{V}{V_o}$
0.00	0.21623	0.00000
0.20	0.21651	0.04326
0.40	0.21736	0.08664
0.60	0.21879	0.13025
0.80	0.22082	0.17420
1.00	0.22346	0.21862
1.20	0.22672	0.26362
1.40	0.23062	0.30934
1.60	0.23515	0.35591
1.80	0.24027	0.40345
2.00	0.24586	0.45205
2.20	0.25163	0.50180
2.40	0.25708	0.55269
2.60	0.26124	0.60455
2.80	0.26255	0.65699
3.00	0.25866	0.70922
3.20	0.24689	0.75993
3.40	0.22545	0.80732
3.60	0.19534	0.84952
3.80	0.16067	0.88516
4.00	0.12658	0.91384
4.20	0.09672	0.93608
4.40	0.07248	0.95291
4.60	0.05370	0.96544
4.80	0.03952	0.97469
5.00	0.02899	0.98149